Analytic calculation of the diffusion coefficient for random walks on strips of finite width: Dependence on size and nature of boundaries

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We study unbiased random walks in discrete time n on a square lattice, in the form of a strip of finite width N in the y direction, with a family of boundary conditions parametrized by a stay probability Γ per time step at the edge sites. The diffusion coefficient $K=\lim_{n\to\infty}\langle X_n^2\rangle/n$ is computed analytically to exhibit its dependence on N and Γ. The result is generalized to the case of a strip with side branches attached to the boundary sites to simulate the effect of rough edges. A further generalization is made to obtain K for a random walk in d dimensions on a lattice bounded in one of the directions. Thus, K serves as a probe of both the transverse size of the region in which diffusion takes place and the nature of the bounding surfaces.

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I. INTRODUCTION

Diffusion in porous and random media [1,2] is a subject of much current interest. Although it is clear that quantities such as the diffusion coefficient, the diffusion front, etc. must carry considerable information about the geometry and the nature of the bounding surfaces of the region in which diffusion takes place, few rigorous and exact results are available. We attempt to investigate such dependences of the diffusion coefficient K in a model that is simple, but in which K can be found analytically: a random walk (RW) on an infinitely long strip of finite width N. Introducing a stay or "sojourn" probability Γ per unit time at sites on the edges of the strip, we have a two-parameter family (N,Γ) characterizing, respectively, the transverse size of the region in which diffusion takes place, and the nature of the boundary. We also generalize our result to obtain exact expressions for K in the case of strips with rough or "spiky" (but not random) boundaries.

The present work is in the general spirit of (but of course far more modest than) other well-known inverse problems, such as that of "hearing the shape of a drum, given perfect pitch" [3–5]. In the latter, one deduces information about the area, perimeter, connectivity, etc. of a two-dimensional region from a knowledge of the spectrum of the Laplacian operator on the region, with Cauchy boundary conditions. (Reference [5] gives an interesting and simpler derivation of many of the key results in terms of a spatially discretized version of the problem.) However, there are several noteworthy differences between this problem and the present work. We are interested (merely) in finding out what information can be obtained from the residue at the *leading* pole of the resolvent of the operator, in a region that is unbounded along at least one direction: the mean-squared displacement in this direction diverges with increasing time and there is, in fact, no nonvanishing equilibrium probability distribution.

Consider a simple unbiased random walk in discrete time n on a square lattice in the form of a strip of width N. The sites are labeled (j,m), where $j\in\mathbb{Z}$ and $m=1,2,\ldots,N$. At every interior site, the walker jumps after a time step to any of the four nearest-neighbor sites with probability $\frac{1}{4}$. At a surface site $(j,1)$ or (j,N), the walker either *remains* at the same site (at the end of a time step) with probability Γ, or jumps to any of its *three* neighbors with probability $(1-\Gamma)/3$ (Fig. 1). The stay probability Γ is a very convenient way of characterizing the nature of the boundary, e.g., its "stickiness," "roughness," etc. The value $\Gamma=\frac{1}{4}$ corresponds to the standard reflecting boundary conditions (on a square lattice). In this case (and in this case *alone*), the motion along the x direction decouples from that along the y direction, in the sense that the probability $P_n(j,m)$ factorizes in the variables j and m for all n, provided one starts with a factorized initial distribution $P_0(j,m)$. The mean-squared displacement $\langle X_n^2\rangle$ in the unbounded direction (the quantity of primary interest here) is exactly equal to $n/2$ in this case, independent of the width N of the strip. This is of course the value of $\langle X_n^2\rangle$ on a square lattice of infinite extent in both directions.

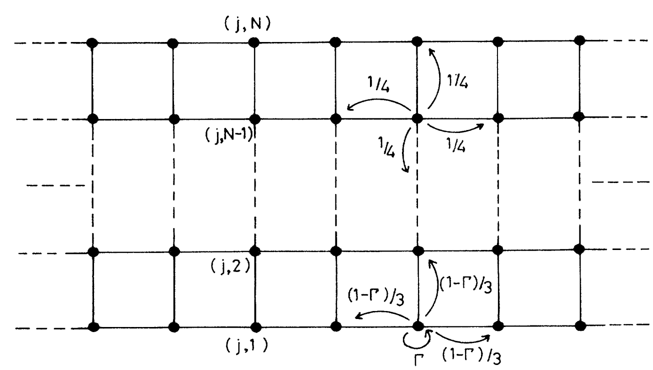

FIG. 1. Lattice of finite width (N sites) in the y direction. Transition probabilities per time step are indicated.

The limiting value $\Gamma=1$ corresponds to fully absorbing boundaries, and there is then no long-range diffusion ($\langle X_n^2\rangle$ does not diverge as $n\to\infty$). At the other extreme, we have $\Gamma=0$—which may be regarded as a "slip" boundary condition. The RW is then a "myopic" one [6]: the walker jumps from every site to any of its available nearest-neighbor sites with equal probability ($\frac{1}{3}$ at the boundary sites, $\frac{1}{4}$ at the interior sites). The motions along the x and y directions are coupled because of the correlation induced by the behavior at the boundaries of the strip. The diffusion coefficient K, defined by the leading asymptotic behavior of $\langle X_n^2\rangle$ according to

$$K=\lim_{n\to\infty}\langle X_n^2\rangle/n\ , \tag{1.1}$$

then becomes N dependent: It has been shown recently [7] that one has, in this case,

$$K=N/(2N-1)\ . \tag{1.2}$$

Thus K depends explicitly on the width N of the strip when $\Gamma=0$, and yields information on the size of the region. We shall first generalize the result to find K for an arbitrary value of Γ, using two different methods. The first is a "direct" calculation in which we derive, as a preliminary step, a useful general formula for K in terms of the discriminant of an $N\times N$ matrix. The second is a calculation in terms of mean first passage times, because this enables us, in turn, to find K for spiky boundaries, in which a side branch emanates from each surface site: As far as diffusion in the x direction is concerned, this situation is equivalent to a RW on a strip without spikes, but with a certain stay probability (to be computed) at the surface sites.

II. DIFFUSION COEFFICIENT ON A STRIP

A. Direct calculation of K

The probability distribution $P_n(j,m)$ obeys the following set of recursion relations:

$$\begin{aligned}P_{n+1}(j,1)&=\Gamma P_n(j,1)+\tfrac{1}{3}(1-\Gamma)[P_n(j-1,1)\\&\qquad+P_n(j+1,1)]\\&\qquad+\tfrac{1}{4}P_n(j,2)\ ,\\P_{n+1}(j,2)&=\tfrac{1}{4}[P_n(j-1,2)+P_n(j+1,2)]\\&\qquad+\tfrac{1}{3}(1-\Gamma)P_n(j,1)+\tfrac{1}{4}P_n(j,3)\ ,\\P_{n+1}(j,m)&=\tfrac{1}{4}[P_n(j-1,m)+P_n(j+1,m)\\&\qquad+P_n(j,m-1)+P_n(j,m+1)]\\&\qquad(3\le m\le N-2)\ ,\end{aligned} \tag{2.1}$$

and two other equations similar to the first two lines for $P_{n+1}(j,N)$ and $P_{n+1}(j,N-1)$, respectively. Next, we define the discrete Fourier and Laplace transform,

$$R(k,m,\xi)=\sum_{n=0}^{\infty}\sum_{j=-\infty}^{\infty}P_n(j,m)e^{ikj}\xi^n\ . \tag{2.2}$$

From the definition of the mean-squared displacement in the x direction, it follows that the quantity we seek is given by

$$\langle X_n^2\rangle=(2\pi i)^{-1}\oint d\xi\,\xi^{-n-1}\times\left[-\frac{\partial^2}{\partial k^2}\sum_{m=1}^{N}R(k,m,\xi)\right]_{k=0}\ , \tag{2.3}$$

where the contour encircles the pole at the origin. As we shall see, the expression in square brackets has a double pole at $\xi=1$. When the integral is evaluated by opening out the contour, the leading asymptotic behavior of $\langle X_n^2\rangle$ comes from the residue at this pole, the factor n [cf. Eq. (1.1)] coming from the derivative of ξ^{-n-1}. The exact determination of $\langle X_n^2\rangle$ is very involved, but its *leading* asymptotic behavior can be extracted more easily. The transform $R(k,m,\xi)\equiv R_m$ obeys the matrix equation $M_{mm'}R_{m'}=f_m$ where $\mathbf{f}$ is specified by the initial distribution $P_0(j,m)=\delta_{j,0}f_m$, and the elements of the $(N\times N)$ tridiagonal matrix $\mathbf{M}$ are as follows:

$$\begin{aligned}&M_{11}=M_{NN}=1-\Gamma\xi-\tfrac{2}{3}(1-\Gamma)\xi\cos k\ ,\\&M_{mm}=1-\frac{\xi}{2}\cos k\quad(2\le m\le N-1)\ ,\\&M_{21}=M_{N-1,N}=-\frac{\xi}{3}(1-\Gamma)\ ,\\&M_{m-1,m}=M_{m+1,m}=-\frac{\xi}{4}\quad(2\le m\le N-1)\ .\end{aligned} \tag{2.4}$$

Then $\mathbf{R}=\mathbf{M}^{-1}\mathbf{f}$, and the sum of its elements is of the form

$$\sum_{m=1}^{N}R(k,m,\xi)=\mathcal{N}(k,\xi)/\Delta(k,\xi)\ , \tag{2.5}$$

where $\mathcal{N}$ is a polynomial in ξ and an entire function of k. The denominator

$$\Delta(k,\xi)=\det\mathbf{M} \tag{2.6}$$

is also a polynomial in ξ and a function of $c=\cos k$ as far as its k dependence is concerned. Moreover, the conservation of probability, $\sum_{j,m}P_n(j,m)=1$ for each $n\ge0$, implies that $\mathcal{N}(0,\xi)/\Delta(0,\xi)=(1-\xi)^{-1}$. Using these facts in Eq. (2.3), we obtain

$$\langle X_n^2\rangle=-(2\pi i)^{-1}\oint d\xi\,\xi^{-n-1}\left[\frac{1}{\Delta}\left[\frac{1}{1-\xi}\frac{\partial\Delta}{\partial c}+\frac{\partial^2\mathcal{N}}{\partial k^2}\right]\right]_{k=0,c=1}\ . \tag{2.7}$$

As each column of $\mathbf{M}$ adds up to $(1-\xi)$ when k is set equal to zero, Δ has a simple zero at $\xi=1$. Therefore it is the first term in square brackets in Eq. (2.7) that has a double pole at $\xi=1$, and is responsible (as we have already explained) for the leading asymptotic behavior of $\langle X_n^2 \rangle$. Writing $\Delta(0,\xi)=(\xi-1)[\partial\Delta(0,\xi)/\partial\xi]_{\xi=1}+\cdots$ and substituting for the factor $1/\Delta$ in Eq. (2.7), we obtain the following compact formula for the diffusion coefficient:

$$K=\left[\frac{\partial\Delta}{\partial c}\Big/\frac{\partial\Delta}{\partial\xi}\right]_{c=1,\xi=1}. \tag{2.8}$$

In practice, the calculation can be simplified further by exploiting the fact that K is independent of the initial condition $P_0(j,m)$. The latter quantity can therefore be chosen so as to maintain the symmetry $P_n(j,m)=P_n(j,N+1-m)$. As a result, one can work with a *reduced* matrix M' of order $[(N+1)/2]$ rather than N that has, however, the same determinant Δ as $\mathbf{M}$. Evaluating the derivatives involved in Eq. (2.8) we find, after all the algebra is done, the result

$$K=\frac{N(1-\Gamma)}{2N(1-\Gamma)+4\Gamma-1}. \tag{2.9}$$

B. K in terms of a mean escape time

Before we discuss the result obtained in Eq. (2.9) above, we digress to point out how K may be obtained by a somewhat simpler method, because it is capable of being generalized to yield K for the more complicated structures to be considered further on.

Let x_n denote the displacement in the horizontal direction *at the nth time step.* The horizontal displacement in n time steps is then $X_n=(x_1+x_2+\cdots+x_n)$. Each x_n can take on the values -1, 0, or $+1$, and is uncorrelated to the others. If the square lattice is *unbounded* in both directions, all its sites are equivalent, and the x_n are identically distributed. This continues to be true on our strip of finite width if perfectly reflecting boundary conditions obtain, i.e., if $\Gamma=\frac{1}{4}$: for, an *unrestricted* RW may be imagined to occur on the unbounded plane comprising the original strip and the infinite set of images formed by successive reflections about its edges. For all other values of Γ, all the sites of the strip are not equivalent (the jump probabilities out of the edge sites are different from those out of the interior sites), so that the x_n are no longer identically distributed random variables. They continue to be uncorrelated to each other, because the RW is Markovian. From Eq. (1.1) it is then clear that

$$K=\lim_{n\to\infty}\left[n^{-1}\sum_{n'=1}^{n}\langle x_{n'}^2\rangle\right]=\lim_{n\to\infty}\langle x_n^2\rangle\equiv\langle x^2\rangle. \tag{2.10}$$

To compute the final quantity on the right, we note that the contribution to $\langle x^2\rangle$ of a step in the vertical direction is zero. As we need the mean-squared displacement in the horizontal direction alone, suppose we project the RW onto a linear chain in the x direction. Excursions in the y direction of the original RW would then correspond to a *stay* at the same site in the projected walk. In effect, therefore, we need the mean-squared displacement *in a single step* for an RW on a linear chain with a certain nonzero stay probability s (at every site) at the end of each time step, corresponding to the walker on the strip going back and forth in the vertical direction without taking a horizontal step. Therefore the probability of a single step in the horizontal direction (for the projected walk) is $1-s$. Hence the mean-squared displacement in a single time step is $\langle x^2\rangle=(1)(1-s)+(0)(s)=1-s$. However, s is easily related to the first moment of the staying probability distribution function at any site on the projected one-dimensional chain: the mean time to escape from any site to either of its neighboring sites on the chain is

$$T=\sum_{n=1}^{\infty}ns^{n-1}(1-s)\Big/\sum_{n=1}^{\infty}s^{n-1}(1-s)=1/(1-s). \tag{2.11}$$

Finally, therefore, the required diffusion coefficient on the original strip is given by

$$K=(1-s)=1/T. \tag{2.12}$$

We note that it is only the mean time of residence T that is required for the determination of K (rather than the full staying probability distribution itself), because we are interested in merely the *leading* asymptotic behavior of $\langle X_n^2\rangle$ as $n\to\infty$; cf. Eq. (2.10). [What we have here is the discrete-time analog of a continuous-time RW with a general nonexponential waiting-time density at each site of the (projected) chain: as long as the first moment τ of this density is finite, the mean-squared displacement $\langle X_n^2(t)\rangle$ has the leading asymptotic behavior t/τ, regardless of the actual form of the waiting-time distribution [8].] These comments apply also to the case of spiky boundaries to be considered in Sec. III. We mention this in order to preclude the impression that Eq. (2.12) for K represents a "mean-field" approximation to K, whereas it is in fact an exact result (both here and for the cases considered in Sec. III).

With reference to the original strip, T is just the mean time to jump from the *set* of sites $\{j_1,m\,|\,m=1,2,\ldots,N\}$ with any given abscissa j_1, to any member of the *set* of sites $\{j_1\pm1,m\,|\,m=1,2,\ldots,N\}$. In other words, T is the mean time to escape from any vertical line $j=j_1$ on the strip to the adjacent lines $j=j_1\pm1$. The problem of finding K reduces to that of finding this T. On an infinite square lattice, translation invariance in the *vertical* direction leads to the equation

$$\tau=\tfrac{1}{2}+\tfrac{1}{2}(1+\tau), \tag{2.13}$$

for the mean time τ to jump in the horizontal direction from any site. Hence $\tau=2$, leading at once to the standard result $K=1/\tau=\frac{1}{2}$, or $\langle X_n^2\rangle=Kn=n/2$. This conclusion is easily seen to be unaltered for a strip with perfectly reflecting boundary conditions. For a general value of Γ (the problem at hand), however, the calculation is less trivial (but still far simpler than the direct calculation given earlier).

C. Evaluation of T

When we geometrically project the strip onto a chain in the x direction, all the N sites (j,m) with a given abscissa j and $1 \le m \le N$ are regarded as *entirely equivalent* sites (or "internal states"); all N of these are collapsed onto a single site on the projected linear chain. We seek T, the mean waiting time between jumps from any site on this chain to a neighboring site on it. On the original strip, let $T_m (m=1,\ldots,N)$ denote the mean first passage time for a walker on the strip starting at a site with ordinate m to take a step in the horizontal direction. As all the N sites with a common abscissa j are equivalent as far as the projection is concerned, the mean waiting time T is just the arithmetic mean of the $\{T_m\}$, i.e.,

$$T=(1/N)\sum_{m=1}^{N} T_m \ . \tag{2.14}$$

It must be noted that there is no m dependent weight factor attached to the T_m in Eq. (2.14), in view of the complete equivalence of the N sites. In particular, T is not the mean of $\{T_m\}$ weighted with any equilibrium distribution, as there is in fact no equilibrium distribution on the infinite lattice.

It is seen readily that $\{T_m\}$ obey the following coupled linear equations:

$$\begin{aligned} T_1 &= 1+\Gamma T_1+\tfrac{1}{3}(1-\Gamma)T_2 \ , \\ T_m &= 1+\tfrac{1}{4}(T_{m-1}+T_{m+1}) \quad (2\le m\le N-1) \ , \\ T_N &= 1+\tfrac{1}{3}(1-\Gamma)T_{N-1}+\Gamma T_N \ . \end{aligned} \tag{2.15}$$

Adding together all the N equations in (2.15), we get

$$\tfrac{1}{2}NT=N+(\Gamma-\tfrac{1}{4})(T_1-\tfrac{1}{3}T_2+T_N-\tfrac{1}{3}T_{N-1}) \ . \tag{2.16}$$

Using the first and the last of Eqs. (2.15) in the second term on the right, we find

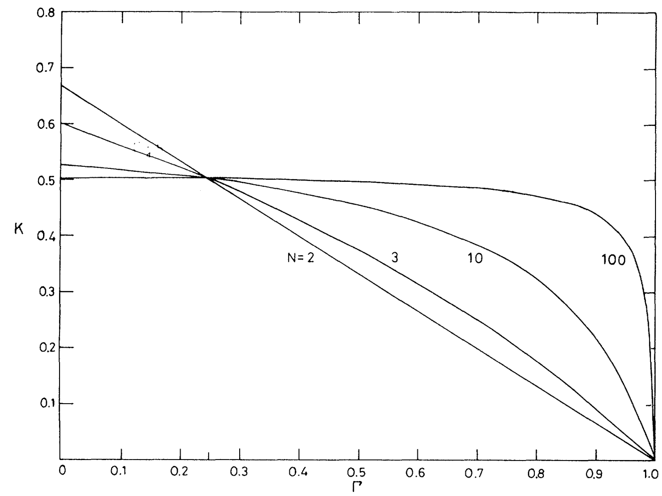

FIG. 2. Variation of the diffusion coefficient K with Γ, the stay probability at boundary sites, for different values of the strip width N. When $\Gamma=\frac{1}{4}$, $K=\frac{1}{2}$ for all $N\ge 2$.

$$T=2+\frac{(4\Gamma-1)}{N(1-\Gamma)} \ . \tag{2.17}$$

On using this in Eq. (2.12), Eq. (2.9) for K follows at once.

Equation (2.9) is easily shown to hold good for $N=2$ as well. This result shows precisely how the diffusion coefficient on a strip depends on both the size of the structure (as specified by the width N) and the nature of the boundary (as characterized by Γ). Figure 2 shows the variation of K with the sojourn probability Γ. We note that $\Gamma=\frac{1}{4}$ (perfectly reflecting boundaries) is a very special, highly "degenerate" case, as stated earlier. The result of Eq. (1.2) for a myopic RW is recovered on setting $\Gamma=0$ in Eq. (2.9).

The foregoing expression is easily generalized to the case of dissimilar boundaries: if the stay probabilities at sites with $m=1$ and N are Γ and Γ', respectively, we find

$$K=\frac{2N(1-\Gamma)(1-\Gamma')}{4N(1-\Gamma)(1-\Gamma')+(4\Gamma-1)(1-\Gamma')+(4\Gamma'-1)(1-\Gamma)} \ . \tag{2.18}$$

III. STRIP WITH SPIKED EDGES

To mimic the effect of "rough" edges, we consider side branches or spikes attached to the edge sites $(j,1)$ and (j,N). Each branch has L sites, and the transition probabilities per time step are as shown in Fig. 3. We note that sites on distinct spikes are not connected directly: to go from one such site to another, the walker *must* pass through the corresponding base sites on the edge of the strip. This makes it possible to write down K for the structure from Eq. (2.12) by the following stratagem. We have merely to compute the mean time of first passage from a site $(j,1)$ on the *edge* of the strip to any of its three neighboring sites *belonging to the strip,* namely, $(j-1,1)$,

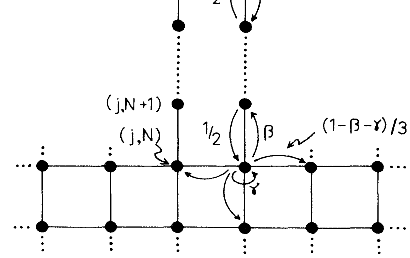

FIG. 3. Strip with identical branches attached to each boundary site. Transition probabilities are indicated.

$(j+1,1)$, or $(j,2)$. In doing so, we must naturally allow for all possible excursions on the spike attached to the site $(j,1)$. The effect of the spike is therefore to produce a certain staying probability distribution at the corresponding base site. As we have already explained, all that is required for the calculation of K is the first moment of this distribution: we must replace the mean exit time $(1-\Gamma)^{-1}$ out of a surface site on the original strip by the mean first passage time referred to above. With this identification, the random walk on the spiked structure becomes equivalent to that on the original structure at the level of the leading asymptotic behavior of $\langle X_n^2\rangle$ (and *only* at this level). A straightforward calculation [9,10] yields, for the required first passage time on the spiked structure, the expression

$$T[(j,1)\to\{(j+1,1),(j-1,1),(j,2)\}]=\frac{2\alpha\beta(L-1)+\alpha+\beta}{\alpha(1-\beta-\gamma)} . \tag{3.1}$$

Substituting the quantity on the right-hand side in Eq. (3.1) for $(1-\Gamma)^{-1}$ in Eq. (2.9), we get, for the diffusion coefficient on the strip with spiked edges,

$$K=\frac{N\alpha(1-\beta-\gamma)}{2(N-2)\alpha(1-\beta-\gamma)+6(L-1)\alpha\beta+3(\alpha+\beta)} . \tag{3.2}$$

As before, this result is easily generalized to the case of dissimilar spikes at the two edges (with parameters α,β,γ,L and $\alpha',\beta',\gamma',L'$, respectively): $(1-\Gamma')^{-1}$ in Eq. (2.18) is replaced by an expression identical to that on the right-hand side in Eq. (3.1), but involving primed quantities.

A number of special cases may be read off from Eq. (3.2). For instance, an unbiased simple RW with reflecting boundary conditions at the tips of the branches implies $\alpha=\frac{1}{2}$, $\beta=\frac{1}{4}$, and $\gamma=0$. Then

$$K=N/(2N+2L) . \tag{3.3}$$

On the other hand, for a myopic RW we have $\alpha=1$, $\beta=\frac{1}{4}$, and $\gamma=0$, leading to

$$K=N/(2N+2L-1) . \tag{3.4}$$

It is evident that this method for finding K is directly applicable if each branch is replaced by any other structure [11,12], as long as the translational invariance in j is retained, and there are no *direct* connections (outside the strip) between the appendages to different sites on the boundary of the original strip. Moreover, the RW should remain diffusive, in the sense that $\langle X_n^2\rangle\sim Kn$ asymptotically. For a random-comb-like situation [13,14], with side branches of random lengths drawn from a common distribution attached to the boundary sites, the foregoing will yield a mean-field expression for K, provided the mean branch length is finite. If this mean is infinite, the behavior of $\langle X_n^2\rangle$ is subdiffusive—signaled, in the present formalism, by the vanishing of K. Extracting the leading asymptotic behavior of $\langle X_n^2\rangle$ in subdiffusive cases is somewhat more involved, and is not our main concern here.

Finally, it is possible to generalize Eq. (2.9) to an RW in $d(\geq 2)$ dimensions on a lattice that is unbounded in $(d-1)$ directions $(x,y,\dots)$, and bounded in any one of the coordinates $(\zeta=1,2,\dots,N)$. By an obvious symmetry we have

$$K=\lim_{n\to\infty}\langle X_n^2\rangle/n=\lim_{n\to\infty}\langle Y_n^2\rangle/n=\cdots . \tag{3.5}$$

As in the case of the strip, the walker at an edge site ($\zeta=1$ or N) has a stay probability Γ per time step, and a probability $(1-\Gamma)/(2d-1)$ of jumping to any of the available nearest-neighbor sites. We then find the result

$$K=\frac{N(1-\Gamma)}{Nd(1-\Gamma)+2\Gamma d-1} . \tag{3.6}$$

When $d=2$, Eq. (2.9) is recovered. The corresponding extensions of Eqs. (2.18), (3.2), etc. follow from Eq. (3.6) in a straightforward manner.

APPENDIX: DIFFUSION COEFFICIENT FOR REGULAR COMBS

As a special case of the method we have used to find K for a strip with spikes, we can write down K in the case of regular combs (linear chains with identical branches) of various kinds (Figs. 4 and 5).

One begins with the basic result $\langle X_n^2\rangle=n$ (i.e., $K=1$) for a simple, unbiased RW on an infinite chain. If the walker *remains* at a site with probability Γ at the end of a time step, and jumps to either one of the two neighboring sites with probability $(1-\Gamma)/2$, then it is shown easily that $\langle X_n^2\rangle=(1-\Gamma)n$, i.e., $K=1-\Gamma$. The mean first passage time out of any site is $(1-\Gamma)^{-1}$. Now consider a chain with branches of *unit* length on either side (Fig. 4), with transition probabilities per unit time as indicated. As the branches are finite in extent, with no absorbing sites, the RW continues to be diffusive. A simple calculation yields the mean first passage time from any backbone site to either of its neighbors on the backbone. Equating this to $(1-\Gamma)^{-1}$ yields the diffusion constant on the comb. The result is

$$K=\alpha\alpha'(1-\beta-\beta'-\gamma)/(\alpha\alpha'+\alpha\beta'+\alpha'\beta) . \tag{A1}$$

A large number of special cases can be read off from (A1). For instance, if the branches occur on one side alone, and the walker does not remain at any site at the end of a time step, we have $\alpha=1$, $\beta'=0$, and $\gamma=0$. Hence $K=(1-\beta)/(1+\beta)$ in this case. If, further, $\beta=\frac{1}{3}$ (the

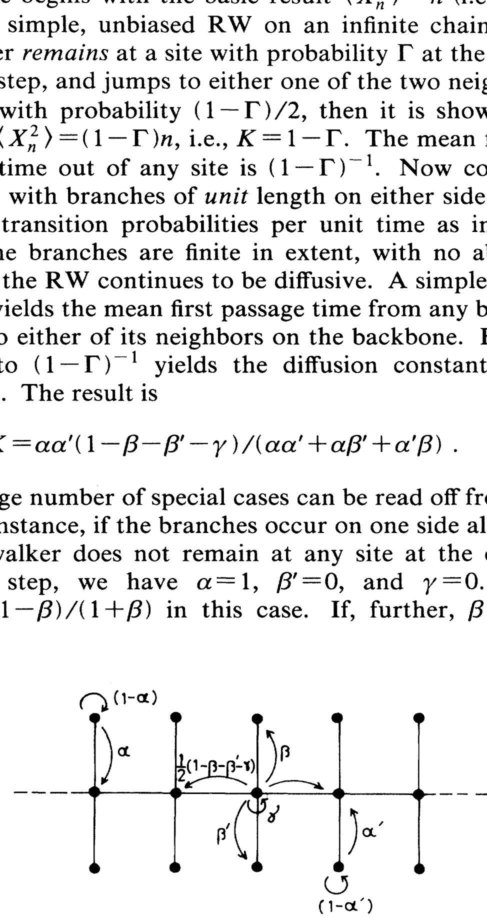

FIG. 4. Chain with branches (of one site each) on either side. Transition probabilities are indicated.

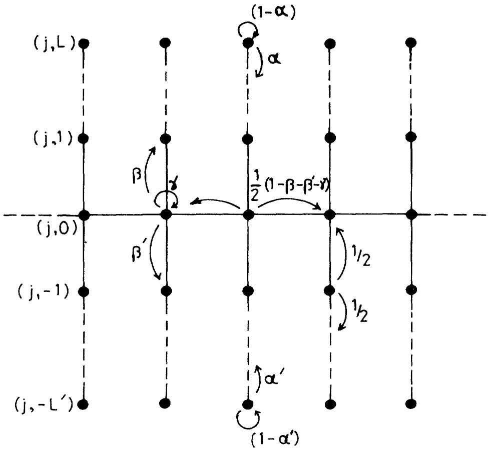

FIG. 5. Chain with branches of L and L' sites, respectively, on the sides. Transition probabilities are indicated.

walker jumps with equal probability to each available nearest-neighbor site), then $K=\frac{1}{2}$ on the chain—the value of K for a simple RW on a square lattice (i.e., $d=2$).

An immediate generalization of Eq. (A1) is possible, to the case of a comb with branches of L and L' sites, respectively (Fig. 5). Since L and L' are finite, the RW is still diffusive. To find the new value of K, all that is needed is to equate the mean first passage time from the first site on the *branch* to the neighboring site on the *backbone* with that for the previous case: in other words, to replace α^{-1} in (A1) by $\alpha^{-1}+2(L-1)$, and α'^{-1} by $\alpha'^{-1}+2(L'-1)$. Therefore

$$K=\frac{\alpha\alpha'(1-\beta-\beta'-\gamma)}{\alpha\alpha'[1+2\beta(L-1)+2\beta'(L'-1)]+\alpha'\beta+\alpha\beta'} . \quad \text{(A2)}$$

Again, a number of special cases may be read off from Eq. (A2). For example, a myopic walker (who jumps from every site to any of the nearest-neighbor sites on the structure with equal probabilities) has $\alpha=\alpha'=1$, $\beta=\beta'=\frac{1}{4}$, and $\gamma=0$: the diffusion constant is, in this case, simply

$$K=1/(L+L'+1) . \quad \text{(A3)}$$

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